

RESTRICTION OF A CHARACTER SHEAF TO CONJUGACY CLASSES

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INTRODUCTION

0.1. Let \mathbf{k} be an algebraically closed field of characteristic $p \geq 0$. Let G be a connected reductive group over \mathbf{k} . Let \hat{G} be the set of (isomorphism classes of) character sheaves on G . (Recall that the character sheaves of G are certain simple perverse sheaves on G (see [L8]) which are equivariant under G -conjugation.) In this paper we are interested in studying the restriction of a character sheaf of G to a conjugacy class of G . For $g \in G$ let $g = g_s g_u = g_u g_s$ be the Jordan decomposition of g (g_s is semisimple, g_u is unipotent). For a subset R of G let $R_s = \{g_s; g \in R\}$, $R_u = \{g_u; g \in R\}$. The set \hat{G} can be naturally partitioned into equivalence classes called *families* as in [L9, 10.6] and (assuming that p is not a bad prime for G) to each family \mathfrak{F} one can attach a unipotent class $C = C_{\mathfrak{F}}$ of G as in [L9, 10.5] so that the following property holds (see [L9, 10.7]).

(a) *If D is a conjugacy class of G such that $\dim D_u \geq \dim C$ and $D_u \neq C$ then $A|_D = 0$ for any $A \in \mathfrak{F}$. There exists a conjugacy class D of G and $A \in \mathfrak{F}$ such that $D_u = C$ and $A|_D \neq 0$.*

Note that (a) characterizes C uniquely. (Actually, in [L9], p is assumed to be sufficiently large or 0 but from this the case where p is only assumed to be not a bad prime can be deduced by standard methods.) If $A \in \hat{G}$ belongs to a family \mathfrak{F} , the unipotent class $C_{\mathfrak{F}}$ is said to be the *unipotent support* of A .

We now state the following refinement of (a).

Theorem 0.2. *Assume that p is not a bad prime for G . Let $\mathfrak{F}, C = C_{\mathfrak{F}}$ be as above. Let D be a conjugacy class of G such that $D_u = C$. Then for any $A \in \mathfrak{F}$ we have $A|_D = \mathcal{L}[\dim(D) + c]$ where \mathcal{L} is a local system on D and $c \in \mathbf{N}$ depends only on A , not on D .*

The proof is given in §1.

0.3. Recall that in [L4, 3.1] we have defined a partition of G into finitely many locally closed smooth irreducible subvarieties of G invariant by conjugation (called

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strata). From the definitions we see that if Y is a stratum of G then Y_u is a single unipotent conjugacy class of G . We have the following result.

Corollary 0.4. *Assume that p is not a bad prime for G . Let A be a character sheaf on G and let Y be a stratum of G . Let \mathfrak{F} be the family in \hat{G} that contains A and let $C = C_{\mathfrak{F}}$.*

- (a) *If $\dim Y_u \geq \dim C$ and $Y_u \neq C$ then $A|_Y = 0$.*
- (b) *If $Y_u = C$ then $A|_Y$ is a local system (up to shift).*

(a) follows immediately from the definition of C . We prove (b). Since all conjugacy classes contained in Y have the same dimension, we see from Theorem 0.2 that there exists $i \in \mathbf{Z}$ such that $\mathcal{H}^j A|_Y = 0$ for $j \neq i$. It remains to show that $\mathcal{H}^i A|_Y$ is a local system. This follows from [L7, 14.2(a)].

0.5. Notation. Let \mathcal{B} be the variety of Borel subgroups of G . We write $\mathcal{B} \times \mathcal{B} = \sqcup_{w \in \mathbf{W}} \mathfrak{D}_w$ where $\mathfrak{D}_w (w \in \mathbf{W})$ are the orbits of G acting on $\mathcal{B} \times \mathcal{B}$ by simultaneous conjugation and \mathbf{W} is the Weyl group (it is naturally a finite Coxeter group with set of simple reflections \mathbf{S}). Let B^* be a fixed Borel subgroup of G and let T^* be a fixed maximal torus of B^* . The parabolic subgroups of G containing B^* are in natural bijection $J \leftrightarrow P_J$ with the subsets $J \subset \mathbf{S}$; thus $P_\emptyset = B^*, P_{\mathbf{S}} = G$. For $J \subset \mathbf{S}$ let L_J be the unique Levi subgroup of P_J such that $T^* \subset L_J$.

If P is a parabolic subgroup of G we denote by U_P the unipotent radical of P and by $\pi_P : P \rightarrow \bar{P} := P/U_P$ the canonical homomorphism. For an affine algebraic group H let H^0 be the identity component of H ; let Z_H be the centre of H . We denote by l a prime number invertible in \mathbf{k} . All sheaves (in particular local systems) are assumed to be $\bar{\mathbf{Q}}_l$ -sheaves. For a group Δ and $g \in \Delta$ let $Z_\Delta(g)$ be the centralizer of g in Δ . If Δ is finite we denote by $\text{Irr}\Delta$ a set of representatives for the isomorphism classes of irreducible representations of Δ over $\bar{\mathbf{Q}}_l$.

If \mathbf{k} is an algebraic closure of the finite field \mathbf{F}_p with p elements and q is a power of p we denote by \mathbf{F}_q the subfield of \mathbf{k} with q elements.

0.6. In §2 we describe a parametrization of the unipotent character sheaves of G (suggested by Theorem 0.2) in terms of restrictions to various conjugacy classes assuming, that p is not a bad prime. This description fails in bad characteristic.

In §3 we establish a canonical bijection between the set of unipotent character sheaves on G and the set of unipotent representations of a split reductive group of the same type over a finite field (here there is no restriction on p). In fact both sets are put in canonical bijection with a combinatorially defined set $\mathfrak{S}_{\mathbf{W}}$ defined purely in terms of the Weyl group \mathbf{W} .

0.7. Now assume that G has connected centre and let \mathcal{L} be a local system of rank 1 on T^* such that $\mathcal{L}^{\otimes m} \cong \bar{\mathbf{Q}}_l$ for some $m \geq 1$ invertible in \mathbf{k} . Let $\hat{G}_{\mathcal{L}}$ be the subset of \hat{G} defined as in [L8, 4.2]. Then the method in §3 can be extended to give a canonical bijection of $\hat{G}_{\mathcal{L}}$ with $\mathfrak{S}_{\mathbf{W}_{\mathcal{L}}}$ where $\mathbf{W}_{\mathcal{L}}$ is the stabilizer of \mathcal{L} in \mathbf{W} .

1. PROOF OF THEOREM 0.2

1.1. A cuspidal pair for G is a pair (Σ, \mathcal{E}) where Σ is the inverse image under $G \rightarrow G/Z_G^0$ of an isolated conjugacy class in G/Z_G^0 and \mathcal{E} is a local system on Σ such that for some $n \geq 1$ invertible in \mathbf{k} ,

\mathcal{E} is equivariant for the $G \times Z_G^0$ action $(g, z) : g_1 \mapsto z^n g g_1 g^{-1}$ on Σ ;

for any parabolic subgroup $P \neq G$ and any $y \in \bar{P}$, we have $H_c^\delta(\pi_P^{-1}(y) \cap \Sigma, \mathcal{E}) = 0$ where $\delta = \dim(\Sigma/Z_G^0) - \dim(\bar{P}) + \dim Z_{\bar{P}}(y)$.

(See [L4, §2]).

An induction datum for G is a triple (L, Σ, \mathcal{E}) where L is a Levi subgroup of a parabolic subgroup of G and (Σ, \mathcal{E}) is a cuspidal pair for L . Given an induction datum (L, Σ, \mathcal{E}) for G we set

$$\Sigma_r = \{g \in \Sigma; Z_G(g_s)^0 \subset L\}$$

$$Y = \sqcup_{x \in G} x \Sigma_r x^{-1}$$

$$\tilde{Y} = \{(g, xL) \in G \times G/L; x^{-1}gx \in \Sigma_r\}$$

$$\hat{Y} = \{(g, x) \in G \times G; x^{-1}gx \in \Sigma_r\}.$$

We have a diagram

$$\Sigma \xleftarrow{\alpha} \hat{Y} \xrightarrow{\beta} \tilde{Y} \xrightarrow{\pi} Y$$

where $\alpha(g, x) = x^{-1}gx$, $\beta(g, x) = (g, xL)$, $\pi(g, xL) = g$. The local system $\alpha^*\mathcal{E}$ on \hat{Y} is L -equivariant for the action of L on \hat{Y} given by $l : (g, x) \mapsto (g, xl^{-1})$ hence it is equal to $\beta^*\tilde{\mathcal{E}}$ for a well defined local system $\tilde{\mathcal{E}}$ on \tilde{Y} . Now $\pi_!\tilde{\mathcal{E}}$ is a well defined local system on Y (a locally closed smooth irreducible subvariety of G). Let $K = IC(\bar{Y}, \pi_!\tilde{\mathcal{E}})$ extended to G by 0 on $G - \bar{Y}$. Here \bar{Y} is the closure of Y in G . We set $f_0 = \dim Y = \dim G - \dim L + \dim \Sigma$. Note that $K[f_0]$ is a perverse sheaf.

1.2. Let $\sigma \in G$ be a semisimple element which is G -conjugate to an element of Σ_s . Let $M = \{x \in G; x^{-1}\sigma x \in \Sigma_s\}$. We have $M \neq \emptyset$. Let $\Gamma = Z_G^0(\sigma) \backslash M/L$, a finite nonempty set. The group $\tilde{\mathcal{W}}_\Sigma := \{y \in G; yLy^{-1} = L, y\Sigma y^{-1} = \Sigma\}$ acts on Γ by $y : \eta \mapsto \eta y^{-1}$. This induces an action of the finite group $\mathcal{W}_\Sigma = \tilde{\mathcal{W}}_\Sigma/L$ on Γ .

For any $x \in M$ let $L_x = xLx^{-1} \cap Z_G^0(\sigma)$; this is a Levi subgroup of a parabolic subgroup of $Z_G^0(\sigma)$. Let $C_x = \{v \in Z_G^0(\sigma); v \text{ unipotent}, x^{-1}\sigma v x \in \Sigma\}$; this is a unipotent class in $Z_G^0(\sigma)$, see [L6, 7.11(c)]. Let $\Sigma_x = Z_{L_x}^0 C_x$. Let \mathcal{E}_x be the local system on Σ_x obtained as the inverse image of \mathcal{E} under the map $\Sigma_x \rightarrow \Sigma$, $g \mapsto x^{-1}\sigma g x$. Then $(\Sigma_x, \mathcal{E}_x)$ is a cuspidal pair for L_x . (See [L6, 7.11(a)].) The definitions in 1.1 are applicable to $Z_G^0(\sigma), L_x, \Sigma_x, \mathcal{E}_x$ instead of $G, L, \Sigma, \mathcal{E}$. Let $\pi_x : tY'_x \rightarrow Y'_x, \tilde{Y}'_x, \tilde{\mathcal{E}}_x, K_x$ be obtained from $\pi : \tilde{Y} \rightarrow Y, \tilde{Y}, \tilde{\mathcal{E}}, K$ in 1.1 by replacing $G, L, \Sigma, \mathcal{E}$ by $Z_G^0(\sigma), L_x, \Sigma_x, \mathcal{E}_x$. We set $f = \dim Y'_x = \dim Z_G^0(\sigma) - \dim L_x + \dim \Sigma_x$; this is independent of the choice of x in M (it is in fact equal to $\dim Z_G^0(\sigma) - \dim L + \dim \Sigma$, see the proof of [L10, 16.10(b)]). Note that $K_x[f]$ is a perverse sheaf on $Z_G^0(\sigma)$ with support equal to \tilde{Y}'_x .

For any $\eta \in \Gamma$ (viewed as a subset of M) we choose a base point $x_\eta \in \eta$. We set

$$\begin{aligned} L_\eta &= L_{x_\eta}, C_\eta = C_{x_\eta}, \Sigma_\eta = \Sigma_{x_\eta}, \pi_\eta = \pi_{x_\eta}, \tilde{Y}'_\eta = \tilde{Y}'_{x_\eta}, \\ Y'_\eta &= Y'_{x_\eta}, \bar{Y}'_\eta = \bar{Y}'_{x_\eta}, \mathcal{E}_\eta = \mathcal{E}_{x_\eta}, \tilde{\mathcal{E}}_\eta = \tilde{\mathcal{E}}_{x_\eta}, K_\eta = K_{x_\eta}. \end{aligned}$$

1.3. Let \mathfrak{U} be the set of all open sets \mathcal{U} in $Z_G^0(\sigma)$ such that

- $1 \in \mathcal{U}$;
- $g\mathcal{U}g^{-1} = \mathcal{U}$ for all $g \in Z_G^0(\sigma)$;
- for $x \in Z_G^0(\sigma)$ we have $x \in \mathcal{U}$ if and only if $x_s \in \mathcal{U}$.

For example, $Z_G^0(\sigma) \in \mathfrak{U}$.

Let $\mathcal{U} \in \mathfrak{U}$. For $\eta \in \Gamma$ we set

$$\tilde{Y}_{\mathcal{U},\eta} = \{(g, xL) \in \tilde{Y}; g \in \sigma\mathcal{U}, x \in \eta\}, Y_{\mathcal{U},\eta} = \pi(\tilde{Y}_{\mathcal{U},\eta}).$$

Let $y \in \mathcal{W}_\Sigma, \eta \in \Gamma$. If $(g, xL) \in \tilde{Y}_{\mathcal{U},\eta}$ then $(g, xy^{-1}L) \in \tilde{Y}_{\mathcal{U},\eta y^{-1}}$ hence $Y_{\mathcal{U},\eta y^{-1}} = Y_{\mathcal{U},\eta}$. Thus for any \mathcal{W}_Σ -orbit Z in Γ we can set $Y_{\mathcal{U},Z} = Y_{\mathcal{U},\eta}$ where η is any element in Z . For Z as above we set $\mathcal{V}_Z = \cap_{\eta \in Z} (\sigma(Y'_\eta \cap \mathcal{U}) \cap Y_{\mathcal{U},Z})$. Let $\mathcal{V} = \cup_Z \mathcal{V}_Z$ where Z runs over the \mathcal{W}_Σ -orbits in Γ . For $\eta \in \Gamma$ let $[\eta]$ be the \mathcal{W}_Σ -orbit of η in Γ .

As shown in [L10, §16] (see also [L6, §8]), for any $\mathcal{U}_1 \in \mathfrak{U}$ we can find $\mathcal{U} \in \mathfrak{U}$ with $\mathcal{U} \subset \mathcal{U}_1$ so that (a),(b),(c) below hold.

(a) \mathcal{V} is an open subset of $Y \cap \sigma\mathcal{U}$ of pure dimension f and the \mathcal{V}_Z (Z runs over the \mathcal{W}_Σ -orbits in Γ) form a finite partition of \mathcal{V} into open and closed subsets; \mathcal{V} is open dense in $\bar{Y} \cap \sigma\mathcal{U} = \cup_{\eta \in \Gamma} \sigma(\tilde{Y}'_\eta \cap \mathcal{U})$; for $\eta \in \Gamma$, $\sigma^{-1}\mathcal{V}_{[\eta]}$ is open dense in $\tilde{Y}'_\eta \cap \mathcal{U}$.

(b) Let ${}^0\tilde{Y} = \{(g, xL) \in \tilde{Y}; g \in \mathcal{V}\}$. For $\eta \in \Gamma$ let ${}^0\tilde{Y}'_\eta = \{(h, zL_\eta) \in \tilde{Y}'_\eta; h \in \sigma^{-1}\mathcal{V}_{[\eta]}\}$. We have a commutative diagram

$$\begin{array}{ccc} \sqcup_{\eta \in \Gamma} {}^0\tilde{Y}'_\eta & \xrightarrow{a} & {}^0\tilde{Y} \\ \downarrow & & \downarrow \\ \sigma^{-1}\mathcal{V} & \xrightarrow{\epsilon} & \mathcal{V} \end{array}$$

where a is the isomorphism given by $(h, zL_\eta) \mapsto (\sigma g, zx_\eta L)$, $\epsilon(h) = \sigma h$ and the vertical maps are given by the first projection.

(c) The canonical isomorphism

$$\oplus_{\eta \in \Gamma} (\pi_{\eta!} \tilde{\mathcal{E}}_\eta)|_{\sigma^{-1}\mathcal{V}_{[\eta]}} \xrightarrow{\sim} \epsilon^*((\pi_! \tilde{\mathcal{E}})|_{\mathcal{V}})$$

of local systems over $\sigma^{-1}\mathcal{V}$ obtained from (b) (where $(\pi_{\eta!} \tilde{\mathcal{E}}_\eta)|_{\sigma^{-1}\mathcal{V}_{[\eta]}}$ is extended by 0 on $\sigma^{-1}\mathcal{V}_{Z'}$ for $Z' \neq [\eta]$) extends uniquely to an isomorphism

$$\kappa : \oplus_{\eta \in \Gamma} K_\eta|_{\mathcal{U}} \xrightarrow{\sim} \epsilon'^*(K|_{\sigma\mathcal{U}})$$

where $\epsilon' : \mathcal{U} \rightarrow \sigma\mathcal{U}$ is $g \mapsto \sigma g$.

(The uniqueness follows from the definition of the intersection cohomology complex.)

Proposition 1.4. *Let E be a semisimple class of G . Let $\hat{E} = \{g \in G; g_s \in E\}$. Let $L, \Sigma, \mathcal{E}, Y, \bar{Y}, K, f_0$ be as in 1.1. Let $c = \dim \mathcal{Z}_L^0$. Then $K|_{\hat{E}}[f_0 - c]$ is a semisimple perverse sheaf.*

Let $\sigma \in E_s$. If $\sigma \notin \bar{Y}_s$ then clearly $K|_{\hat{E}} = 0$. Thus we can assume that $\sigma \in \bar{Y}_s$. It follows that σ is as in 1.2. Let Ξ be the set of unipotent elements in $Z_G^0(\sigma)$. Note that $\sigma\Xi \subset \hat{E}$. It is enough to show that $K[f - c]|_{\sigma\Xi}$ is a semisimple perverse sheaf where f is as in 1.2. (Note that $\dim(G/Z_G^0(\sigma)) = f_0 - f$.) With notation of 1.3(c) we have

$$(a) \quad \bigoplus_{\eta \in \Gamma} K_\eta|_{\Xi} \xrightarrow{\sim} \epsilon''^*(K|_{\sigma\Xi})$$

where $\epsilon'' : \Xi \rightarrow \sigma\Xi$ is $g \mapsto \sigma g$. (We use that $\Xi \subset \mathcal{U}$.) We see that it is enough to show that for any $\eta \in \Gamma$, $K_\eta|_{\Xi}[f - c]$ is a semisimple perverse sheaf. We define a local system \mathcal{E}_η^1 on $\Sigma_\eta = \mathcal{Z}_{L_\eta}^0 C_\eta = \mathcal{Z}_L^0 C_\eta$ (C_η as in 1.2) as $\bar{\mathbf{Q}}_l \boxtimes (\mathcal{E}_\eta|_{1 \times C_\eta})$ where $\bar{\mathbf{Q}}_l$ is viewed as a local system on \mathcal{Z}_L^0 . We define K_η^1 in terms of $Z_G^0(\sigma), L_\eta, \Sigma_\eta, \mathcal{E}_\eta^1$ in the same way as K was defined in terms of $G, L, \Sigma, \mathcal{E}$ in 1.1. From [L10, 15.2] we see that there is a canonical isomorphism

$$(b) \quad (K_\eta)|_{\Xi} \xrightarrow{\sim} (K_\eta^1)|_{\Xi}.$$

It is then enough to show that $K_\eta^1|_{\Xi}[f - c]$ is a semisimple perverse sheaf. But this follows from [L4, (6.6.1)]. The proposition is proved.

1.5. Remark. In the special case where $E = \{1\}$, $L = \Sigma$ is a maximal torus of G , $\mathcal{E} = \bar{\mathbf{Q}}_l$, the proposition above reduces to a result in [BM].

Corollary 1.6. *Let A be a character sheaf of G and let E, \hat{E} be as in 1.4. Then for some integer m , $A|_{\hat{E}}[m]$ is a semisimple perverse sheaf.*

We can find $L, \Sigma, \mathcal{E}, Y, \bar{Y}, K$ as in 1.1 such that A is a direct summand of $K[f_0]$ (f_0 as in 1.1). Hence $A|_{\hat{E}}[-c]$ is a direct summand of $K|_{\hat{E}}[f_0 - c]$ which by 1.4 is a semisimple perverse sheaf. It follows that $A|_{\hat{E}}[-c]$ is itself a semisimple perverse sheaf.

1.7. We prove Theorem 0.2. Let $A \in \mathfrak{F}$. Let $C = C_{\mathfrak{F}}$. We can find $L, \Sigma, \mathcal{E}, Y, \bar{Y}, K$ as in 1.1 such that A is a direct summand of $K[f_0]$ (f_0 as in 1.1). We can also assume that \mathcal{E} is irreducible. Let $\sigma \in D_s$. If $\sigma \notin \bar{Y}_s$ then clearly $K|_D = 0$ hence $A|_D = 0$. Thus we can assume that $\sigma \in \bar{Y}_s$. It follows that σ is as in 1.2. Let Ξ be the set of unipotent elements in $Z_G^0(\sigma)$. By the proof of Proposition 1.4 and with the notation there we have

$$\epsilon''^*(K|_{\sigma\Xi})[f - c] \cong \bigoplus_{\eta \in \Gamma} K_\eta^1[f - c]|_{\Xi} \cong A_1 \oplus A_2 \oplus \dots \oplus A_n$$

where A_1, \dots, A_n are simple perverse sheaves on Ξ , equivariant under conjugation by $Z_G^0(\sigma)$. Since $\epsilon''^*(A|_{\sigma\Xi})[f - c - f_0]$ is a direct summand of $\epsilon''^*(K|_{\sigma\Xi})[f - c]$ it

follows that $\epsilon''^*(A|_{\sigma\Xi})[f - c - f_0]$ is isomorphic to a direct summand of $A_1 \oplus A_2 \oplus \dots \oplus A_n$ hence

$$(a) \quad \epsilon''^*(A|_{\sigma\Xi})[f - c - f_0] \cong \bigoplus_{j \in J} A_j \text{ for some } J \subset [1, n].$$

Now for each $j \in J$ there is a unique unipotent class \mathfrak{c}_j of $Z_G^0(\sigma)$ such that $A_j|_{\mathfrak{c}_j}$ is of the form $\mathcal{L}_j[\dim \mathfrak{c}_j]$ where \mathcal{L}_j is an irreducible local system and $A_j|_{\mathfrak{c}'} = 0$ for any unipotent class \mathfrak{c}' of $Z_G^0(\sigma)$ such that $\mathfrak{c}' \not\subset \overline{\mathfrak{c}_j}$. It follows that for any unipotent class \mathfrak{c} of $Z_G^0(\sigma)$ we have

$$(b) \quad \epsilon''^*(A|_{\sigma\mathfrak{c}})[f - c - f_0] \cong \bigoplus_{j \in J; \mathfrak{c} \subset \overline{\mathfrak{c}_j} - \mathfrak{c}_j} A_j|_{\mathfrak{c}} \oplus \bigoplus_{j \in J; \mathfrak{c} = \mathfrak{c}_j} \mathcal{L}_j[\dim \mathfrak{c}].$$

Now assume that \mathfrak{c} is a unipotent class of $Z_G^0(\sigma)$ such that $\mathfrak{c} \subset C$. Assume that

$$(c) \quad \bigoplus_{j \in J; \mathfrak{c} \subset \overline{\mathfrak{c}_j} - \mathfrak{c}_j} A_j|_{\mathfrak{c}} \neq 0$$

that is, there exists $j_0 \in J$ such that $\mathfrak{c} \subset \overline{\mathfrak{c}_{j_0}} - \mathfrak{c}_{j_0}$, $A_{j_0}|_{\mathfrak{c}} \neq 0$. Using (a) we deduce

$$\epsilon''^*(A|_{\sigma\mathfrak{c}_{j_0}})[f - c - f_0] \cong \bigoplus_{j \in J} A_j|_{\mathfrak{c}_{j_0}}.$$

The last direct sum contains as a summand $A_{j_0}|_{\mathfrak{c}_{j_0}} = \mathcal{L}_{j_0}[\dim \mathfrak{c}_{j_0}] \neq 0$ hence the whole direct sum is $\neq 0$ and $\epsilon''^*(A|_{\sigma\mathfrak{c}_{j_0}})[f - c - f_0] \neq 0$. Thus $A|_{\sigma\mathfrak{c}_{j_0}} \neq 0$. Let $\tilde{\mathfrak{c}}$ be the unipotent class of G that contains \mathfrak{c}_{j_0} . Since $\mathfrak{c} \subset \overline{\mathfrak{c}_{j_0}} - \mathfrak{c}_{j_0}$, we see that $C \subset \tilde{\mathfrak{c}} - \tilde{\mathfrak{c}}$. If D' is the conjugacy class of G that contains $\sigma\mathfrak{c}_{j_0}$ we have $D'_u = \tilde{\mathfrak{c}}$, $\dim D'_u > \dim C$. Hence by 0.1(a) we have $A_{D'} = 0$ so that $A|_{\sigma\mathfrak{c}_{j_0}} = 0$, a contradiction. Thus the assumption (c) cannot hold so that the direct sum in (c) is zero and (b) reduces to

$$\epsilon''^*(A|_{\sigma\mathfrak{c}})[f - c - f_0] \cong \bigoplus_{j \in J; \mathfrak{c} = \mathfrak{c}_j} \mathcal{L}_j[\dim \mathfrak{c}].$$

We see that $A|_{\sigma\mathfrak{c}}[f - c - f_0 - \dim \mathfrak{c}]$ is a local system. Applying this to \mathfrak{c} equal to one of $\mathfrak{c}_1, \mathfrak{c}_2, \dots, \mathfrak{c}_r$, the unipotent classes in $Z_G^0(\sigma)$ that are contained in C and satisfy $\{g \in D; g_s = \sigma\} = \sigma\mathfrak{c}_1 \sqcup \sigma\mathfrak{c}_2 \sqcup \dots \sqcup \sigma\mathfrak{c}_r$ we deduce that $A|_{\sigma\mathfrak{c}_1 \sqcup \dots \sqcup \sigma\mathfrak{c}_r}[f - c - f_0 - d]$ is a local system. (Here $d = \dim \mathfrak{c}_1 = \dots = \dim \mathfrak{c}_r$; note that $\mathfrak{c}_1, \dots, \mathfrak{c}_r$ are permuted transitively by $Z_G(\sigma)$ hence have the same dimension). It also follows that $A|_D[f - c - f_0 - d]$ is a local system. We have $f_0 - f + d = \dim(G/Z_G^0(\sigma)) + d = \dim(D)$. Hence $A|_D[-\dim(D) - c]$ is a local system. Theorem 0.2 is proved.

1.8. The following result can be proved in the same way as Theorem 0.2 with no restriction on p .

Let $A \in \hat{G}$. Assume that C_0 is a unipotent class of G such that the following holds: if D is a conjugacy class of G such that $\dim D_u \geq \dim C_0$ and $D_u \neq C_0$ then $A|_D = 0$. Then for any conjugacy class D of G such that $D_u = C$ we have $A|_D = \mathcal{L}[\dim(D) + c]$ where \mathcal{L} is a local system on D and $c \in \mathbf{N}$ depends only on A , not on D .

1.9. The local system \mathcal{L} which appears in Theorem 0.2 can be reducible. For example if G is of type B_2 (resp. G_2) and p is not a bad prime for G then there exist character sheaves on G with unipotent support equal to the subregular unipotent class C in G whose restriction to C is (up to shift) the direct sum of $\bar{\mathbf{Q}}_l$ and an irreducible local system of rank 1 (resp. 2).

2. UNIPOTENT CHARACTER SHEAVES

2.1. For $w \in \mathbf{W}$ let $Y_w = \{(g, B) \in G \times \mathcal{B}; (B, gBg^{-1}) \in \mathfrak{D}_w\}$ and let $\pi_w : Y_w \rightarrow G$ be the first projection. Let \hat{G}^{un} be the subset of \hat{G} consisting of unipotent character sheaves that is, those character sheaves on G which appear as constituents of the perverse cohomology sheaf ${}^p H^i(\pi_{w!} \bar{\mathbf{Q}}_l)$ for some $w \in \mathbf{W}, i \in \mathbf{Z}$. Note that \hat{G}^{un} is a union of families of \hat{G} . In fact, as in [L8, 4.6], we have a partition $\hat{G}^{un} = \sqcup_{\mathcal{F}} \hat{G}_{\mathcal{F}}^{un}$ where \mathcal{F} runs over the families in $\text{Irr} \mathbf{W}$ and for each family \mathcal{F} of $\text{Irr} \mathbf{W}$, $\hat{G}_{\mathcal{F}}^{un}$ is a family of \hat{G} .

If $A \in \hat{G}^{un}$ the proof of Theorem 0.2 for A simplifies somewhat. In this case Proposition 1.4 is only needed in the case where (L, C, \mathcal{E}) satisfies the condition that \mathcal{E} is equivariant for the $L \times Z_L^0$ -action $(g, z) : g_1 \mapsto zgg_1g^{-1}$ on Σ (hence \mathcal{E} is the inverse image under $\Sigma \rightarrow \Sigma/Z_L^0$ of an irreducible local system on Σ/Z_L^0). Hence if σ is as in 1.2 then with notation in the proof of 1.4, for $\eta \in \Gamma$, the local system \mathcal{E}_η on Σ_η is equivariant for the action of $Z_{L_\eta}^0 = Z_L^0$ on Σ_η (left multiplication) hence $\mathcal{E}_\eta = \mathcal{E}_\eta^1$ and $K_\eta^1 = K_\eta$. Thus the step 1.4(b) in the proof of 1.4 is unnecessary in this case.

We shall try to make Theorem 0.2 more precise in the case of unipotent character sheaves. In the remainder of this section we assume that p is not a bad prime for G .

2.2. For a finite group Δ , $M(\Delta)$ is the set of all pairs (g, ρ) where $g \in \Delta$ is defined up to conjugacy and $\rho \in \text{Irr} Z_\Delta(g)$.

2.3. In the remainder of this section we fix a family \mathcal{F} of $\text{Irr} \mathbf{W}$. Let $\mathfrak{F} = \hat{G}_{\mathcal{F}}^{un}$ be the corresponding family in \hat{G}^{un} . In this case $C = C_{\mathfrak{F}}$ is the special unipotent class of G such that the corresponding Springer representation of \mathbf{W} belongs to \mathcal{F} . Let \mathcal{S}_C be the set of conjugacy classes D in G such that $D_u = C$. Let $\omega \in C$. Let $A(\omega) = Z_G(\omega)/Z_G^0(\omega)$ and let $\Delta = \bar{A}(\omega)$ be the canonical quotient of $A(\omega)$ defined in [L3, (13.1.1)]. Let $Z_G(\omega) \xrightarrow{j'} A(\omega) \xrightarrow{h} \Delta$ be the obvious (surjective) homomorphisms; let $j = hj' : Z_G(\omega) \rightarrow \Delta$. Let $[\Delta]$ be the set of conjugacy classes in Δ . For $D \in \mathcal{S}_C$ let $\phi(D)$ be the conjugacy class of $j(g_s)$ in Δ where $g \in D$ is such that $g_u = \omega$; clearly such g exists and is unique up to $Z_G(\omega)$ -conjugacy so that the conjugacy class of $j(g_s)$ is independent of the choice of g . Thus we get a (surjective) map $\phi : \mathcal{S}_C \rightarrow [\Delta]$. For $\gamma \in [\Delta]$ we set $\mathcal{S}_{C,\gamma} = \phi^{-1}(\gamma)$. We now select for each $\gamma \in [\Delta]$ an element $x_\gamma \in \gamma$. Let $D \in \mathcal{S}_{C,\gamma}$, $\mathcal{E} \in \text{Irr} Z_\Delta(x_\gamma)$. We can find $g \in D$ such that $g_u = \omega, j(g_s) = x_\gamma$ (and another choice for such g must be of the form bgb^{-1} where $b \in Z_G(\omega), j(b) \in Z_\Delta(x_\gamma)$). Let \mathcal{E}^D be the G -equivariant local system on D whose stalk at $g_1 \in D$ is $\{z \in G; zgz^{-1} = g_1\} \times \mathcal{E}$ modulo the equivalence relation $(z, e) \sim (zh^{-1}, j(h)e)$ for all $h \in Z_G(g)$. If g is changed to $g_1 = bgb^{-1}$ (b as above) then \mathcal{E}^D is changed to the G -equivariant local system \mathcal{E}_1^D on D whose stalk at $g' \in D$ is $\{z' \in G; z'g_1z'^{-1} = g'\} \times \mathcal{E}$ modulo the equivalence relation $(z', e') \sim (z'h'^{-1}, j(h')e')$ for all $h' \in Z(g_1)$. We have an isomorphism of local systems $\mathcal{E}^D \xrightarrow{\sim} \mathcal{E}_1^D$ which for any $g' \in D$ maps the stalk

of \mathcal{E}^D at g' to the stalk of \mathcal{E}_1^D at g' by the rule $(z, e) \mapsto (zb^{-1}, j(b)e)$. (We have $zb^{-1}g_1bz^{-1} = zgz^{-1} = g'$.) This is compatible with the equivalence relations. Thus the isomorphism class of the local system \mathcal{E}^D does not depend on the choice of g .

Using the methods sketched in [L12, §4] one can prove the following refinement of Theorem 0.2.

Theorem 2.4. (a) Let $A \in \hat{G}_{\mathcal{F}}^{un}$. There exists a unique $\gamma \in [\Delta]$ and a unique $\mathcal{E} \in \text{Irr}Z_{\Delta}(x_{\gamma})$ such that

- (i) if $D \in \mathcal{S}_{C, \gamma}$, we have $A|_D \cong \mathcal{E}^D[\dim(D) + c]$ (c as in 1.7).
- (ii) if $D \in \mathcal{S}_{C, \gamma'}$ with $\gamma' \in [\Delta] - \{\gamma\}$, we have $K|_D = 0$.
- (b) The assignment $A \mapsto (\gamma, \mathcal{E})$ in (a) defines a bijection $\hat{G}_{\mathcal{F}}^{un} \xrightarrow{\sim} M(\Delta)$.

3. PARAMETRIZATION OF UNIPOTENT REPRESENTATIONS AND OF UNIPOTENT CHARACTER SHEAVES

3.1. Let (W, S) be a Weyl group. Let $n = |S|$. Let ν_W be the number of reflections of W . For any subset J of S we denote by W_J the subgroup of W generated by J . Assuming that W is irreducible or $\{1\}$ we define a set \mathfrak{S}_W^0 as follows.

If $W = \{1\}$ we have $\mathfrak{S}_W^0 = \{1\}$.

If W is of type A_n ($n \geq 1$) we have $\mathfrak{S}_W^0 = \emptyset$.

If W is of type B_n or C_n ($n \geq 2$) we have $\mathfrak{S}_W^0 = \{(-1)^{n/2}\}$ if $n = k^2 + k$ for some integer $k \geq 1$ and $\mathfrak{S}_W^0 = \emptyset$, otherwise.

If W is of type D_n $n \geq 4$ we have $\mathfrak{S}_W^0 = \{(-1)^{n/4}\}$ if $n = 4k^2$ for some integer $k \geq 1$ and $\mathfrak{S}_W^0 = \emptyset$, otherwise.

If W is of type E_6 then $\mathfrak{S}_W^0 = \{\zeta \in \bar{\mathbf{Q}}_l; (\zeta^3 - 1)/(\zeta - 1) = 0\}$.

If W is of type E_7 then $\mathfrak{S}_W^0 = \{\zeta \in \bar{\mathbf{Q}}_l; (\zeta^4 - 1)/(\zeta^2 - 1) = 0\}$.

If W is of type E_8 then $\mathfrak{S}_W^0 = \{\zeta \in \bar{\mathbf{Q}}_l; (\zeta^4 - 1)(\zeta^5 - 1)(\zeta^6 - 1)/(\zeta^2 - 1) = 0\}$ with $\zeta = 1$ appearing twice as $\zeta = 1'$ and $\zeta = 1''$.

If W is of type F_4 then $\mathfrak{S}_W^0 = \{\zeta \in \bar{\mathbf{Q}}_l; (\zeta^2 - 1)(\zeta^3 - 1)(\zeta^4 - 1)/(\zeta^2 - 1) = 0\}$ with $\zeta = 1$ appearing twice as $\zeta = 1'$ and $\zeta = 1''$.

If W is of type G_2 then $\mathfrak{S}_W^0 = \{\zeta \in \bar{\mathbf{Q}}_l; (\zeta^2 - 1)(\zeta^3 - 1)/(\zeta - 1) = 0\}$.

The exponents 4, 5, 6 which appear in type E_8 satisfy: $4 = n/2$, $5 \times 6 = \text{Coxeter number of } W$, $4 \times 5 \times 6 = \nu_W$.

The exponents 2, 3, 4 which appear in type F_4 satisfy: $2 = n/2$, $3 \times 4 = \text{Coxeter number of } W$, $2 \times 3 \times 4 = \nu_W$.

If W is not irreducible or $\{1\}$, we define $\mathfrak{S}_W^0 = \mathfrak{S}_{W_1}^0 \times \dots \times \mathfrak{S}_{W_r}^0$ where $W = W_1 \times \dots \times W_r$ with W_1, \dots, W_r irreducible Weyl groups ($r \geq 2$).

Assume now that W is irreducible or $\{1\}$. If $J \subset S$ and $\mathfrak{S}_{W_J}^0 \neq \emptyset$ then W_J is irreducible or $\{1\}$ and for any $J' \subset S$ such that $J \subset J'$, conjugation by the longest element $w_0^{J'}$ of $W_{J'}$ leaves J stable; using [L1, 5.9] it follows that the involutions $\sigma_h := w_0^{J \cup h} w_0^J = w_0^J w^{J \cup h}$ ($h \in S - J$) generate a subgroup of W which is itself a Weyl group denoted by $W^{S/J}$. For $h \neq h'$ in $S - J$ the order of $\sigma_h \sigma_{h'}$ is equal to

$$2(\nu_{W_{J \cup h \cup h'}} - \nu_{W_J})/(\nu_{W_{J \cup h}} + \nu_{W_{J \cup h'}} - 2\nu_{W_J}).$$

$W^{S/J}$ is a complement to W_J in the normalizer of W_J in W .) Let \mathfrak{S}_W be the set of all triples (J, ϵ, ζ) where $J \subset S$, $\epsilon \in \text{Irr} W^{S/J}$, $\zeta \in \mathfrak{S}_{W_J}^0$. We have an imbedding $\mathfrak{S}_W^0 \rightarrow \mathfrak{S}_W$, $\zeta \mapsto (S, 1, \zeta)$. (Note that $W^{S/S} = \{1\}$.)

If W is not irreducible or $\{1\}$, we define $\mathfrak{S}_W = \mathfrak{S}_{W_1} \times \dots \times \mathfrak{S}_{W_r}$ where $W = W_1 \times \dots \times W_r$ with W_1, \dots, W_r irreducible Weyl groups ($r \geq 2$).

3.2. Now assume that \mathbf{k} is an algebraic closure of the finite field \mathbf{F}_p with p elements and that G has a fixed \mathbf{F}_q -split rational structure with Frobenius map $F : G \rightarrow G$. We also assume that $F(B^*) = B^*$, $F(T^*) = T^*$. We fix a square root \sqrt{q} of q in $\bar{\mathbf{Q}}_l$. Now F induces a map $F : \mathcal{B} \rightarrow \mathcal{B}$. For any $w \in \mathbf{W}$ let $X_w = \{B \in \mathcal{B}; (B, F(B)) \in \mathfrak{D}_w\}$ (see [DL]). Now G^F acts on X_w by conjugation hence there is an induced action of G^F on $H_c^i(X_w, \bar{\mathbf{Q}}_l)$ for $i \in \mathbf{Z}$. Also X_w is stable under $F : \mathcal{B} \rightarrow \mathcal{B}$ hence the linear map $F^* : H_c^i(X_w, \bar{\mathbf{Q}}_l) \rightarrow H_c^i(X_w, \bar{\mathbf{Q}}_l)$ is well defined for any i . For any $\mu \in \bar{\mathbf{Q}}_l^*$ let $H_c^i(X_w, \bar{\mathbf{Q}}_l)_\mu$ be the generalized μ -eigenspace of this linear map; it is a G^F -submodule of $H_c^i(X_w, \bar{\mathbf{Q}}_l)$. Let \mathfrak{U}_q be the set of all $\rho \in \text{Irr} G^F$ such that ρ appears in the G^F -module $H_c^i(X_w, \bar{\mathbf{Q}}_l)$ for some $w \in \mathbf{W}, i \in \mathbf{Z}$. (This is the set of unipotent representations of G^F .) For any $\rho \in \mathfrak{U}_q$ and any $w \in \mathbf{W}$ we denote by $(\rho : R_w)$ the multiplicity of ρ in the virtual representation $\sum_i (-1)^i H_c^i(X_w, \bar{\mathbf{Q}}_l)$.

According to [L2, 3.9] for any $\rho \in \mathfrak{U}_q$ there is a well defined coset $\tilde{\tau}_\rho$ of $\bar{\mathbf{Q}}_l^*$ modulo its subgroup $\{q^r; r \in \mathbf{Z}\}$ such that whenever ρ appears in the G^F -module $H_c^i(X_w, \bar{\mathbf{Q}}_l)_\mu$ (with $i \in \mathbf{Z}, w \in \mathbf{W}, \mu \in \bar{\mathbf{Q}}_l^*$) we have $\mu \in \tilde{\tau}_\rho$; now $\tilde{\tau}_\rho$ is contained in a unique coset (denoted by τ_ρ) of $\bar{\mathbf{Q}}_l^*$ modulo $\{\sqrt{q}^r; r \in \mathbf{Z}\}$.

Let \mathfrak{U}_q^0 be the set of all $\rho \in \mathfrak{U}_q$ which are cuspidal. We have the following result. (In the case where \mathbf{W} is of type E_8 (resp. F_4) we denote by w_* an element of \mathbf{W} whose characteristic polynomial in the reflection representation \mathbf{W} is $(X^4 - X^2 + 1)^2$ (resp. $(X^2 - X + 1)^2$).

Theorem 3.3. *Assume that G/\mathcal{Z}_G is simple or $\{1\}$. There exists a unique bijection $\mathfrak{S}_{\mathbf{W}}^0 \xrightarrow{\sim} \mathfrak{U}_q^0$ with the following properties. If $\rho \in \mathfrak{U}_q^0$ corresponds to $\zeta \in \mathfrak{S}_{\mathbf{W}}^0$ then $\zeta \in \tau_\rho$. If in addition G/\mathcal{Z}_G is of type E_8 or F_4 and ρ' (resp. ρ'') in \mathfrak{U}_q^0 corresponds to $1'$ (resp. $1''$) in $\mathfrak{S}_{\mathbf{W}}^0$ then $(\rho' : R_{w_*}) = 1$ and $(\rho'' : R_{w_*}) = 0$.*

This follows from [L3, 11.2] and its proof.

3.4. Now let $J \subset \mathbf{S}$ and let ρ_0 be a unipotent cuspidal representation of L_J^F . Then L_J/\mathcal{Z}_{L_J} is simple or $\{1\}$ hence by 3.3 applied to L_J , ρ_0 corresponds to an element $\zeta \in \mathfrak{S}_{W_J}^0$. Let $I(J, \zeta)$ be the representation of G^F induced by ρ_0 viewed as a representation of P_J^F . This is a direct sum of irreducible representations in \mathfrak{U}_q ; the set of all $\rho \in \mathfrak{U}_q$ which appear in $I(J, \zeta)$ is denoted by $\mathfrak{U}_{q,J,\zeta}$. According to [L2, 3.26], the set $\mathfrak{U}_{q,J,\zeta}$ is in natural bijection with the set of irreducible representations of a Hecke algebra (a deformation of the Weyl group $\mathbf{W}^{S/J}$ in 3.1 with parameters being powers of q explicitly described in [L2, p.35]); hence it is in natural bijection with the set $\text{Irr} \mathbf{W}^{S/J}$ (here we use our choice of \sqrt{q}). We have the following result.

Corollary 3.5. *Assume that G/\mathcal{Z}_G is simple or $\{1\}$. There exists a unique bijection $\mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \mathfrak{U}_q$ with the following property. If $\rho \in \mathfrak{U}_q$ corresponds to $(J, \epsilon, \zeta) \in$*

$\mathfrak{S}_{\mathbf{W}}$ then $\rho \in \mathfrak{U}_{q,J,\zeta}$ and ρ corresponds as above to $\epsilon \in \text{Irr } \mathbf{W}^{\mathbf{S}/J}$.

3.6. We now drop the assumption on \mathbf{k} made in 3.2. Let A be a simple perverse sheaf on G which is equivariant for the conjugation G -action on G . We define an invariant $\lambda_A \in \bar{\mathbf{Q}}_l^*$ (a root of 1) as follows. We can find an open dense subset N of the support of A which is invariant under conjugation by G and an irreducible G -equivariant local system \mathcal{L} on N such that $A|_N = \mathcal{L}[\delta]$, $\delta = \dim N$. If $g \in N$ then $Z_G(g)$ acts naturally and irreducibly on the stalk \mathcal{L}_g . Since g is in the centre of $Z_G(g)$, it acts on \mathcal{L}_g as a nonzero scalar which is independent of the choice of g and is denoted by λ_A . In particular λ_A is defined for any character sheaf A on G . (I have found this definition of λ_A in the late 1980's (unpublished); it was also found later in [E]. The definition makes sense also when G is replaced by a finite group Γ and A is replaced by an irreducible vector bundle on Γ equivariant for the conjugation Γ -action; in this case the definition appears in [L3, 11.1] and it inspired my later definition for character sheaves.)

Let $\hat{G}^{0,un}$ be the set of all character sheaves in \hat{G}^{un} (see 2.1) which are cuspidal. For $A \in \hat{G}^{un}$ and $w \in \mathbf{W}$ we set

$$(A : \mathcal{K}_w) = \sum_{i \in \mathbf{Z}} (-1)^{\dim G + i} (\text{multiplicity of } A \text{ in } {}^p H^i(\pi_{w!} \bar{\mathbf{Q}}_l))$$

where $\pi_{w!}$ is as in 2.1. We have the following result (with w_* as in 3.2).

Theorem 3.7. *Assume that G/Z_G is simple or $\{1\}$. There exists a unique bijection $\mathfrak{S}_{\mathbf{W}}^0 \xrightarrow{\sim} \hat{G}^{0,un}$ with the following properties. If $A \in \hat{G}^{0,un}$ corresponds to $\zeta \in \mathfrak{S}_{\mathbf{W}}^0$ then $\zeta = \lambda_A$. If in addition G/Z_G is of type E_8 or F_4 and A' (resp. A'') in $\hat{G}^{0,un}$ corresponds to $1'$ (resp. $1''$) in $\mathfrak{S}_{\mathbf{W}}^0$ then $(A' : \mathcal{K}_{w_*}) = 1$ and $(A'' : \mathcal{K}_{w_*}) = 0$.*

This follows from the known classification of unipotent cuspidal character sheaves and the known formulas for the numbers $(A : \mathcal{K}_w)$ (when $p = 2$, the cleanness results of [L11] must be used).

3.8. Now let $J \subset \mathbf{S}$ and let A_0 be a unipotent cuspidal character sheaf on L_J . Then L_J/Z_{L_J} is simple or $\{1\}$ hence by 3.7 applied to L_J , A_0 corresponds to an element $\zeta \in \mathfrak{S}_{\mathbf{W}_J}^0$. The semisimple perverse sheaf $\text{ind}_{P_J}^G A_0$ on G (see [L5, 4.1, 4.3]) is a direct sum of unipotent character sheaves on G ; the set of all $A \in \hat{G}^{un}$ which appear in $\text{ind}_{P_J}^G A_0$ is denoted by $\hat{G}_{J,\zeta}^{un}$ and is in canonical bijection with the set of isomorphism classes of simple modules of $\text{End}(\text{ind}_{P_J}^G A_0)$ hence with $\text{Irr } \mathbf{W}^{\mathbf{S}/J}$ (one can show that the algebra $\text{End}(\text{ind}_{P_J}^G A_0)$ is canonically isomorphic to the group algebra of $\mathbf{W}^{\mathbf{S}/J}$).

Corollary 3.9. *Assume that G/Z_G is simple or $\{1\}$. There exists a unique bijection $\mathfrak{S}_{\mathbf{W}} \xrightarrow{\sim} \hat{G}^{un}$ with the following property. If $A \in \hat{G}^{un}$ corresponds to $(J, \epsilon, \zeta) \in \mathfrak{S}_{\mathbf{W}}$ then $A \in \hat{G}_{J,\zeta}^{un}$ and A corresponds as above to $\epsilon \in \text{Irr } \mathbf{W}^{\mathbf{S}/J}$.*

3.10. We now return to the setup in 3.2. We assume that G/\mathcal{Z}_G is simple or $\{1\}$. Combining the bijections in 3.5, 3.9 we obtain a canonical bijection $\hat{G}^{un} \leftrightarrow \mathfrak{U}_q$. Here are some properties of this bijection. If $A \in \hat{G}^{un}$ corresponds to $\rho \in \mathfrak{U}_q$ then $\lambda_A \in \tau_\rho$; moreover $(A : \mathcal{K}_w) = (\rho : R_w)$ for any $w \in W$.

3.11. Now assume that \mathbf{k} is as in 3.2 and that G has a fixed \mathbf{F}_p -split rational structure with Frobenius map $F_0 : G \rightarrow G$. We also assume that $F_0(B^*) = B^*$, $F_0(T^*) = T^*$. We fix a square root \sqrt{p} of p in $\bar{\mathbf{Q}}_l$. For any integer $t > 0$, $F_0^t : G \rightarrow G$ is the Frobenius map for an \mathbf{F}_{p^t} -rational structure on G so that the definitions in 3.2 are applicable with $q = p^t$ and F_0^t instead of F . (We set $\sqrt{q} = (\sqrt{p})^t$.) For $w \in W$ we write $X_{w,t}$ instead of X_w of 3.2 (thus $X_{w,t} = \{B \in \mathcal{B}; (B, F_0^t(B)) \in \mathfrak{D}_w\}$). We want to show that in a certain sense the set \hat{G}^{un} is the limit of the sets \mathfrak{U}_{p^t} as t tends to 0.

For $w \in \mathbf{W}$ let $Y_{w,t} = \{(g, B) \in G \times \mathcal{B}; (B, gF_0^t(B)g^{-1}) \in \mathfrak{D}_w\}$ and let $\pi_{w,t} : Y_{w,t} \rightarrow G$ be the first projection. Now G acts on $Y_{w,t}$ by $x : (g, B) \mapsto (xgF_0^t(x^{-1}), xBx^{-1})$ and on G (transitively) by $x : g \mapsto xgF_0^t(x^{-1})$; then $\pi_{w,t}$ is compatible with the G -actions. Let \hat{G}_t^{un} be the set of isomorphism classes of irreducible perverse sheaves on G which appear as constituents of the perverse cohomology sheaf ${}^pH^i((\pi_{w,t})^*\bar{\mathbf{Q}}_l)$ for some $w \in \mathbf{W}, i \in \mathbf{Z}$. If $A \in \hat{G}_t^{un}$ then A is G -equivariant for a transitive G -action on G hence it is of the form $\mathcal{L}[\dim G]$ where \mathcal{L} is an irreducible G -equivariant local system on G . Since the isotropy group at 1 of the G -action on G is $G^{F_0^t}$ we see that \mathcal{L} is completely determined by the (irreducible) representation of $G^{F_0^t}$ on the stalk at 1 of \mathcal{L} . This irreducible representation is clearly unipotent and we thus obtain a bijection $\hat{G}_t^{un} \leftrightarrow \mathfrak{U}_{p^t}$. (Note that ${}^pH^i((\pi_{w,t})^*\bar{\mathbf{Q}}_l)$ is up to shift the G -equivariant local system on G such that the isotropy group $G^{F_0^t}$ acts on the stalk at 1 as on the $G^{F_0^t}$ -module $H_c^{i'}(X_{w,t}, \bar{\mathbf{Q}}_l)$ for some i' .) On the other hand if we now take $t = 0$ then F_0^t becomes the identity map and $Y_{w,t}, \pi_{w,t}$ become Y_w, π_w in 2.1. The definition of \hat{G}_t^{un} specializes to the definition of \hat{G}^{un} . In this sense the set \hat{G}^{un} can be viewed as limit of the sets \mathfrak{U}_{p^t} as t tends to 0.

For $w \in W$ we define a map $\Phi_t : Y_{w,t} \rightarrow Y_{w,t}$ by $\Phi_t(g, B) = (g, gF_0^t(B)g^{-1})$. This induces for any i an isomorphism of perverse sheaves ${}^pH^i((\pi_{w,t})^*\bar{\mathbf{Q}}_l) \rightarrow {}^pH^i((\pi_{w,t})^*\bar{\mathbf{Q}}_l)$. For any $\mu \in \bar{\mathbf{Q}}_l^*$ we denote by ${}^pH^i((\pi_{w,t})^*\bar{\mathbf{Q}}_l)_\mu$ the generalized μ -eigenspace of the last isomorphism. From the definitions we see that if $\tau \in \mathfrak{U}_{p^t}$ corresponds as above to $A \in \hat{G}_t^{un}$ then we have $\mu \in \tau_\rho$ whenever A is a constituent of ${}^pH^i((\pi_{w,t})^*\bar{\mathbf{Q}}_l)_\mu$ for some w, i .

When t is replaced by 0, the cosets of $\bar{\mathbf{Q}}_l^*$ modulo its subgroup $\{\sqrt{p}^{tr}; r \in \mathbf{Z}\}$ become just the elements of $\bar{\mathbf{Q}}_l^*$ and for $w \in W$ the map $\Phi_t : Y_{w,t} \rightarrow Y_{w,t}$ becomes the map $\Phi_0 : Y_w \rightarrow Y_w$ given by $(g, B) \mapsto (g, gBg^{-1})$. This induces for any i an isomorphism of perverse sheaves ${}^pH^i(\pi_w^*\bar{\mathbf{Q}}_l) \rightarrow {}^pH^i(\pi_w^*\bar{\mathbf{Q}}_l)$ denoted again by Φ_t . For any $\mu \in \bar{\mathbf{Q}}_l^*$ we denote by ${}^pH^i(\pi_w^*\bar{\mathbf{Q}}_l)_\mu$ the generalized μ -eigenspace of the last isomorphism. One can check from the definitions that if $A \in \hat{G}^{un}$ is a constituent

of ${}^p H^i(\pi_{w!} \bar{\mathbf{Q}}_l)_\mu$ then $\mu = \lambda_A$. On the other hand λ_A can be viewed as the limit as t tends to 0 of the coset τ_ρ where $\rho \in \mathfrak{U}_{p^t}$ corresponds to A .

3.12. Now assume that G is the identity component of a possibly disconnected reductive group over \mathbf{k} and that G^1 is a connected component of that reductive group. Then the notion of character sheaf on G^1 is well defined (see [L8].) Let A be a character sheaf on G^1 . Then to A we can associate a root of 1 $\lambda_A \in \bar{\mathbf{Q}}_l^*$ as in 3.6. More precisely let d be an integer ≥ 1 such that $g^d \in G$ for any $g \in G^1$. Note that A is a simple perverse sheaf on G^1 , G -equivariant for the conjugation action of G on G^1 . Let N be an open dense subset of the support of A which is invariant by G -conjugation and is such that $A|_N$ is up to shift a local system \mathcal{L} . If $g \in N$ then the centralizer of g in G acts naturally and irreducibly on the stalk \mathcal{L}_g . Since g^d is in the centre of this centralizer it acts on \mathcal{L}_g as a nonzero scalar which is independent of the choice of N and is denoted by λ_A .

Using the invariant λ_A we can find a combinatorial parametrization of the set of unipotent character sheaves on G^1 parallel to that in 3.9.

Similarly, the set of unipotent representations of a not necessarily \mathbf{F}_q -split connected group over \mathbf{F}_q can be parametrized in the same spirit using the invariant defined in [L2, 3.9].

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